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New Connection between Spinors and Geometry

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We study those nonlinear infinitesimal realizations of $SL(2,C)$ that leave invariant the quadratic function $\dot{x}_\mu \dot{x}_\mu$ of the four-velocity components of a particle. These transformations are defined as maps of a larger manifold, which includes the four-velocity space, into itself in such a way that transformations of the $\dot{x}_\mu$ depend upon other functions in the manifold. The requirement that $\dot{x}_\mu \dot{x}_\mu$ remain invariant limits the types of other functions that can contribute in the transformation of the $\dot{x}_\mu$. However, among those allowed are the spinors and a three-dimensional space that transforms nonlinearly and recently associated with electric charge. We point out and explore two interesting aspects of these nonlinear realizations. First, they generally necessitate interactions since $\dot{x}_\mu = 0$ is not a covariant equation. Second, with superposition of solutions, exact measurement of the four-velocity or space-time position, is impossible. This and related features of nondeterministic measurement inherent to these realizations are discussed.

1. INTRODUCTION

The invariant metric form

$$g^{\mu\nu}\dot{x}_\mu \dot{x}_\nu$$

where $\dot{x}_\mu \equiv dx_\mu/ds$ with $\dot{x}_4 = ic(dt/ds)$, seems to characterize rather well our physical space-time. Over local regions, the approximations $g^{\mu\nu} \approx \delta_{\mu\nu}$ is consistent with a broad spectrum of observations, so that the local geometry may be characterized by the invariant form

$$\dot{x}_\mu \dot{x}_\mu$$

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This form remains invariant under the usual linear realization of the six-parameter Lorentz group. In these transformations, the four-velocity manifold V^4, is mapped into itself.

The purpose of this paper is to describe certain nonlinear realizations of the Lorentz group [actually we consider $SL(2, C)$, the two fold covering group] which leave $\dot{x}_\mu \dot{x}_\mu$ invariant. These transformations are defined as a map of a larger manifold, which includes V^4, into itself in such a way that the transformations on the $\dot{x}_\mu$ have a nontrivial dependence on other functions in the manifold. That such realizations exist for the Lorentz group is not surprising since similar nonlinear realizations are well known for other groups.[2]

After considering in Section 2 the general expressions for the commutator equations for these realizations, we discuss the form of these equations for the special cases in which we have (a) superposition of solutions, and (b) realizations that reduce to the linear case in some limit. In this class are realizations for which the nonlinearity (deviation from linear realizations) is characterized by product terms of the form $\psi_i \phi_j$ where both ψ_i and ϕ_j are components from two separate six-dimensional spaces. The ϕ_j transform like the components of the electromagnetic field. On the other hand the ψ_j cannot transform in this way. The equations which the ψ_j must satisfy restrict the way they can transform.

In Section 3 we study in detail two very different basic realizations. In one case the ψ_j are related directly to a four-component spinor. In the second, the ψ_j transform with a particular nonlinear pattern that cannot be linearized; and recently associated via minimal coupling with electric charge (Dalton, 1982a, 1982b). In addition to these two basic solutions we briefly consider a coupled solution involving them.

Because the nonlinear transformations have a local space-time dependence, zero acceleration equations of the form $\ddot{x}_\mu = 0$, are generally not covariant. On the other hand, we show (in Section 4) that equations with the well-known Lorentz force form $\ddot{x}_\mu = F_{\mu\nu} \dot{x}_\nu$.are covariant under the nonlinear transformations. Thus, an important characteristic of these realizations is that they necessitate accelerations or interactions (Dalton, 1980a, b). In the limit that the nonlinearity vanishes and the transformations reduce to linear ones, those interactions can vanish.

Under these nonlinear transformations a vector $\dot{x}$ in V^4 is transformed into a vector $\dot{x}'$ which differs from a vector that would be obtained from $\dot{x}$ if one assumed the usual linear transformation. From this one can see that a submanifold in V^4 with points connected by a nonlinear realization will

[2]See, for instance, Weinberg (1968). In this study chirality transformations on the nucleon field have a dependence on the pion field.

generally differ from a submanifold in V^4 with points connected by a linear realization even though $\dot{x}_\mu \dot{x}_\mu$ is left invariant in both cases. In Section 4 we discuss this point and the idea that in macroscopic averages over many particles, the nonlinearity could average to zero. This raises a serious question in general. How can one use any macroscopic experimental apparatus which transforms linearly to make *deterministic* measurements of quantities that transform nonlinearly? The conclusion in this paper is that you cannot make covariant deterministic measurements of quantities that transform nonlinearly using experimental apparatus that transforms linearly.

Another problem of measurement also arises in this theory. Assuming superposition of solutions, for the ψ_j, deviation from linear realizations will depend upon contributions from all spinor and other fields in a region. In this case, the field of a probe particle will change the transformation properties of a particle one wishes to detect.

2. INFINITESIMAL TRANSFORMATIONS AND COMMUTATOR RELATIONS

In this section we consider linear and nonlinear infinitesimal transformations of the group $G = SL(2, C)$, the twofold covering group of the Lorentz group, as it acts simultaneously on the four-velocity components $\dot{x}_\mu$ and other functions of physical interest. We describe the general commutator relations and recall some properties of linear transformations for contrast with the nonlinear ones. We also describe the restrictions imposed by the commutator relations in order to have superposition of solutions, as well as a general class of solutions that contain the linear case as a limit. Discussion of explicit solutions are relegated to Section 3.

We first consider here some notation convenient for the sequel. The symbols α, β, γ will represent sets of six group parameters, that is, $\alpha = \{\alpha_i | i = 1, \ldots, 6\}$. If ε is a typical variable in the manifold on which the group acts, then for an infinitesimal transformation corresponding to element $g(\alpha) \in G$ we use the following notation:

$$\begin{aligned} g(\alpha)\colon\ \varepsilon \to \varepsilon' &= F(\alpha, \{\varepsilon\}) \\ &= \varepsilon + \alpha_i \left. \frac{\partial F(\beta, \{\varepsilon\})}{\partial \beta_i} \right|_{\beta = 0} \\ &= \varepsilon + \alpha_i (\delta_i \varepsilon) \end{aligned} \tag{1}$$

Here, we have used $\{\varepsilon\}$ to represent the set of all variables including ε on which G simultaneously acts. We also use the convention unless specified otherwise, of summing from one to six over repeated Latin indices (these indices indicate the six group parameters). In the second line of (1) we have made a Taylor expansion (for small α_i) and used $F(0,\{\varepsilon\}) \equiv \varepsilon$. In the third line we used the expression $(\delta_i \varepsilon)$ which we in general define as follows:

$$(\delta_i \varepsilon) \equiv \left. \frac{\partial F(\beta, \{\varepsilon\})}{\partial \beta_i} \right|_{\beta=0} \tag{2}$$

The use of the parenthesis in $(\delta_i \varepsilon)$ helps avoid confusion in the composite expression discussed below.

Using the above notation we now discuss the basic commutator equations. If we evaluate the infinitesimal transformation corresponding to the product $g(\alpha)g(\beta)g(\alpha^{-1})g(\beta^{-1})$ and impose closure we arrive at the following general commutator relation (for a derivation see Appendix A of Dalton, 1982a):

$$\big(\delta_i(\delta_j \varepsilon)\big) - \big(\delta_j(\delta_i \varepsilon)\big) = C_{ijk}(\delta_k \varepsilon) \tag{3}$$

In this equation the $C_{ijk} = -C_{jik}$ are the group structure constants (these constants must satisfy the Jacobi identities which are coupled algebraic equations imposed by the associative nature of the group product). With both i and j ranging over the six group parameters, (3) represents a system of 15 equations which the transformations in (1), whether linear or nonlinear, must satisfy.

Before considering nonlinear transformations we recall for contrast some features of the more familiar linear transformations of G on the four-velocities $\dot{x}_\mu$. For these we have

$$\dot{x}'_\mu = \dot{x}_\mu + \alpha_i(\delta_i \dot{x}_\mu) \tag{4}$$

where

$$(\delta_i \dot{x}_\mu) = -S_i^{\mu\rho}\dot{x}_\rho \tag{5}$$

$$(\delta_j S_i^{\mu\rho}) = 0 \tag{6}$$

If we put (5) with (6) into (3), we get the following commutator relations (expressed in matrix form):

$$[S_i, S_j] \equiv S_i S_j - S_j S_i = C_{ijk} S_k \tag{7}$$

The matrix elements in (5) are further restricted by the imposition that $\dot{x}_\mu \dot{x}_\mu$ remains invariant under the transformation (4). Explicitly we impose $(\delta_i(\dot{x}_\mu \dot{x}_\mu)) = 0$, which means

$$\begin{aligned}(\delta_i(\dot{x}_\mu \dot{x}_\mu)) &= 2\dot{x}_\mu(\delta_i \dot{x}_\mu) \\ &= 2\dot{x}_\mu S_i^{\mu\rho} \dot{x}_\rho \\ &= 0 \end{aligned} \tag{8}$$

This expression is true provided

$$S_i^{\mu\rho} = -S_i^{\rho\mu} \tag{9}$$

With regard to the nonlinear transformations discussed below there are some related points of interest in the above linear transformations. First, (6) is not necessary in order for (8) to be valid. Second, the conventional commutator relations in (7) are not valid unless (6) holds.

We now turn our discussion to some nonlinear infinitesimal realizations of G for which $\dot{x}_\mu \dot{x}_\mu$ is left invariant. We still consider transformations which have the form given in (5), that is,

$$(\delta_i \dot{x}_\mu) = -M_i^{\mu\rho} \dot{x}_\rho \tag{10}$$

but in this case we do not impose a condition like (6) on the $M_i^{\mu\rho}$. Now since $(\delta_i M_j^{\mu\rho}) \neq 0$ we see from the relation

$$\dot{x}'_\mu = \dot{x}_\mu - \alpha_i M_i^{\mu\rho} \dot{x}_\rho \tag{11}$$

that $\dot{x}'_\mu$ is at least a quadratic function of variables on which G acts. The matrix elements $M_i^{\mu\rho}$ in (11) may depend on any of the variables in $\{\varepsilon\}$ including the $\dot{x}_\mu$. In the case where $(\delta_i M_j^{\mu\rho}) = 0$ the transformation in equation (11) reduces to the linear case discussed above. If we put (10) into the general commutator equation (3) we arrive at the following expression (written in matrix form):

$$(\delta_i M_j) - (\delta_j M_i) + [M_i, M_j] = C_{ijk} M_k \tag{12}$$

From this expression we can see that we get a relation like (7) for the M_i if and only if $(\delta_i M_j) = 0$ for all i and j.

Now if we impose the condition that (11) leave invariant the form $\dot{x}_\mu \dot{x}_\mu$ we can follow the steps given in (8) to arrive at the condition

$$M_i^{\mu\rho} = -M_i^{\rho\mu} \tag{13}$$

for elements of all six matrices M_i. With (13) each M_i has only six independent elements. As a consequence, one may use the six S_i discussed above in the linear case to write the M_i in the following expanded form:

$$M_i = -\zeta_{ij} S_j \tag{14}$$

The 36 one-dimensional functions ζ_{ij} $(i, j = 1, \ldots, 6)$ in (14) may be functions of any variable (even the x_μ) on which the group acts.

If we insert (14) into (11) using a Lie Algebra basis with real parameters α, and require that the x_i (x_4) remain real (pure imaginary) under this transformation, then we must impose the condition

$$\zeta_{ij} = \zeta_{ij}^* \tag{15}$$

where * means complex conjugate. We will return later to the implications of this restriction for particular solutions.

Using (14) in (12) leads after some algebra to the following equation for the 36 ζ_{ij}:

$$(\delta_i \zeta_{jl}) - (\delta_j \zeta_{il}) - \zeta_{ik} \zeta_{jm} C_{kml} = C_{ijk} \zeta_{kl} \tag{16}$$

As pointed out in Dalton (1980a), solutions of this particular set of equations have been studied previously for both internal and external (space-time) symmetry groups. Perhaps the most widely known study of realizations like these is the $SU(2) \times SU(2)$ chirality realization in which the ζ_{ij} for that group were functions of pion fields (Weinberg, 1968).

Given two solutions ε_{ij} and $\bar{\varepsilon}_{ij}$ of (16) we now look at the conditions for which the superposition of these two solutions is also a solution. Putting

$$\zeta_{ij} = \varepsilon_{ij} + \bar{\varepsilon}_{ij} \tag{17}$$

into (16) leads to the following algebraic relation:

$$(\varepsilon_{ik} \bar{\varepsilon}_{jm} + \bar{\varepsilon}_{ik} \varepsilon_{jm}) C_{kml} = 0 \tag{18}$$

Equation (18) represents algebraic relations between the ε_{ik} and $\bar{\varepsilon}_{ik}$ which must be satisfied in order for their sum to be a solution. In (Dalton, 1980a) several different solutions of (16) were described, among which there were pairs of solutions which satisfied the superposition condition (18). We note here that there may be compositions of solutions other than (17), such as products, which also satisfy (16), but discussion of these is outside the scope of this article.

One solution of (16) is $\zeta_{ij} = -\delta_{ij}$ for which the realizations reduce to the linear ones. If solutions are to have the linear case as a limit, then the expression for ζ_{ij} should have the factor δ_{ij} in one term. The other special solution of (15) is the null solution $\zeta_{ij} = 0$.

We now look at the general class of solutions of (16) which has the linear limit $-\delta_{ij}$ as one term. For these solutions we write ζ_{ij} in the following form:

$$\zeta_{ij} = -\delta_{ij} + \eta_{ij} \tag{19}$$

where it is understood that η_{ij} does not include a term like δ_{ij}. Using (19) in (16) produces the following equation:

$$(\delta_i \eta_{jl}) - (\delta_j \eta_{il}) + \eta_{jm} C_{iml} + \eta_{ik} C_{kjl} - \eta_{ik} \eta_{jm} C_{kml} = C_{ijk} \eta_{kl} \tag{20}$$

Equation (20) simplifies for those cases where the η_{ik} factorize as follows:

$$\eta_{ik} = \psi_i \phi_k \tag{21}$$

where ψ_i and ϕ_k are functions whose transformation properties must satisfy (20) which we rearrange as follows:

$$\begin{aligned} &\left[(\delta_i \psi_j) - (\delta_j \psi_i) - C_{ijk} \psi_k\right] \phi_l + \psi_j \left[(\delta_i \phi_l) + \phi_m C_{iml}\right] \\ &\quad - \psi_i \left[(\delta_j \phi_l) + \phi_m C_{jml}\right] = 0 \end{aligned} \tag{22}$$

This equation is satisfied if we have the following separate relations.

$$(\delta_i \psi_j) - (\delta_j \psi_i) = C_{ijk} \psi_k \tag{23}$$

$$(\delta_i \phi_l) = -C_{iml} \phi_m \tag{24}$$

It is interesting to note that the particular set of equations in (23) was obtained in a recent study of diagonal nonlinear realizations. The local nature of these realizations necessitated the interaction of minimal coupling (Dalton, 1982a, b).

3. SOLUTIONS

In this section we discuss realizations of $SL(2, C)$ which satisfy (23) and (24) above. In particular we consider a spinor solution of (23) as well as one nonlinear solution that recently has been associated with the Coulomb potential of a point charge.

We first consider (24). There are six functions ϕ_i, which in the second Lie Algebra basis of the Appendix are indicated by three ϕ_j^+ and three ϕ_j^- (here $j=1,2,3$). In this basis (24) may be expressed as follows:

$$(\delta_i^+\phi_j^+) = -\varepsilon_{ijk}\phi_k^+ \tag{25}$$

$$(\delta_i^-\phi_j^-) = -\varepsilon_{ijk}\phi_k^- \tag{26}$$

$$(\delta_i^+\phi_j^-) = 0, \qquad (\delta_i^-\phi_j^+) = 0 \tag{27}$$

If we use these expressions to evaluate the Casimir invariants (A11) and (A12), we find that ϕ_j^+ and ϕ_j^- transform respectively under the (1,0) and (0,1) representations of $SL(2,C)$ (Dalton, 1980b).

For the combinations $\phi_j = (\phi_j^+ + \phi_j^-)/2$ and $\hat{\phi}_j = -i(\phi_j^+ - \phi_j^-)/2$ we also consider equation (24) in the first Lie Algebra basis of the Appendix:

$$(\delta_i\phi_j) = -\varepsilon_{ijk}\phi_k \tag{28}$$

$$(\delta_i\hat{\phi}_j) = -\varepsilon_{ijk}\hat{\phi}_k \tag{29}$$

$$(\hat{\delta}_i\phi_j) = -\varepsilon_{ijk}\hat{\phi}_k \tag{30}$$

$$(\hat{\delta}_i\hat{\phi}_j) = +\varepsilon_{ijk}\phi_k \tag{31}$$

From these equations we can see that if ϕ_j and $\hat{\phi}_j$ are real, their transformed values will also be real. We will recall this point below where we discuss the physical constraint (15). If ϕ_j and $\hat{\phi}_j$ are real, then ϕ_j^- will be the complex conjugate of ϕ_j^+. This is the case for the electromagnetic field. In comparison with the latter, the ϕ_i above transform like components of the magnetic field **B**, and the $\hat{\phi}_i$ above transform like components of the electric field **E**.

We now consider solutions of (23). Since there are six ψ one might at first glance expect the ψ_i to transform under the (1,0) or (0,1) representation (like the ϕ_i above). However, as pointed out in Dalton (1982a) these realizations do not solve (23). This is because equation (23) represents conditions on the antisymmetric parts of $(\delta_i\psi_j)$. To see this we write $(\delta_i\psi_j) = S_{ij} + A_{ij}$, where $S_{ij} = S_{ji}$ is the symmetric part of $(\delta_i\psi_j)$, and $A_{ij} = -A_{ji}$ is the antisymmetric part. Using this in (23) we find

$$A_{ij} = \tfrac{1}{2}C_{ijk}\psi_k \tag{32}$$

Thus provided $(\delta_j\psi_i)$ is not zero (23) determines uniquely the antisymmetric part of $(\delta_i \psi_j)$, with the 1/2 factor as indicated in (32). By contrast, the expressions in (25) and (26) do not have a factor of 1/2 as in (32). With (32) we can write $(\delta_i \psi_j)$ as follows:

$$(\delta_i \psi_j) = S_{ij} + C_{ijk} \psi_k / 2 \tag{33}$$

The symmetric part of S_{ij} in (33) is not arbitrary, because the $(\delta_i \psi_j)$ must also satisfy the general commutator equation (3).

For more detailed discussion we now consider (23) in the first basis of the Appendix:

$$(\delta_i \psi_j) - (\delta_j \psi_i) = -\varepsilon_{ijk} \psi_k \tag{34}$$

$$(\delta_i \hat{\psi}_j) - (\hat{\delta}_j \psi_i) = -\varepsilon_{ijk} \hat{\psi}_k \tag{35}$$

$$(\hat{\delta}_i \hat{\psi}_j) - (\hat{\delta}_j \hat{\psi}_i) = +\varepsilon_{ijk} \psi_k \tag{36}$$

The reader is reminded that we have one function (ψ_k or $\hat{\psi}_k$) for each group parameter. Through common indices the three ψ_k are associated with the SU(2) subgroup parameters and the three $\hat{\psi}_k$ are associated with the three pure Lorentz boost.

In order to satisfy (15) the three ψ_j and three $\hat{\psi}_j$ must be real, assuming that the six ϕ_k were chosen real. [Pure imaginary ϕ_k, ψ_j, and $\hat{\psi}_j$ would also satisfy (15)].

The factor of 1/2 in (33) and the requirement that the ψ_j and $\hat{\psi}_j$ be real are nontrivial constraints on these realizations. One solution has been given in Dalton (1982a), and because of its interesting connection with the Coulomb potential of a point charge we briefly recall its properties here. For this realization we have $\psi_j = 0$ and

$$(\hat{\delta}_i \hat{\psi}_j) = \pm \hat{\psi} \delta_{ij} \mp \hat{\psi}_i \hat{\psi}_j / \hat{\psi} \tag{37}$$

where $\hat{\psi}^2 = \hat{\psi}_i \hat{\psi}_i$ is an invariant in this particular realization. This particular pattern of nonlinear transformation has been reviewed by Philip and Wigner (1968) for the de Sitter group. If X_u and Y_u indicate the position of coordinates of a field and source point in Minkowski space with $r_u = X_\mu - Y_\mu$ then the expression

$$\hat{\psi}_i = F(r) r_i \tag{38}$$

will satisfy (37) if and only if $r_u r_u = 0$ and $F = q/r$ where q is some constant. The two signs in (37) correspond to the retarded and advanced signals from the source to the field point. In the classification scheme discussed in the appendix, $\hat{\psi}_j$ transforms as a (0,0) realization. The zero values for j_1 and j_2 are due to the particular nonlinear nature of the transformation. In Dalton (1982a), q was associated with the constant in the minimal coupling potential A_μ (for the transverse gauge $A_4 = 2q/r$). This minimal coupling interaction was generated by a local nonlinear transformation on the wavefunction.

We now consider spinor solutions [that is, (1/2,0) and (0,1/2) linear realizations] of (23). To satisfy the condition (15) with real ϕ_j we consider ψ_j and $\hat{\psi}_j$ which are related to the real and imaginary parts of functions q_j as follows:

$$\psi_j = \left(q_j + q_j^*\right)/2 \tag{39}$$

$$\hat{\psi}_j = -i\left(q_j - q_j^*\right)/2 \tag{40}$$

where * indicates complex conjugation. By construction, ψ_j and $\hat{\psi}_j$ are real. If we use these expressions in (34)–(36) we arrive at the following relations:

$$\left(\delta_i{}^+ q_j\right) - \left(\delta_j{}^+ q_i\right) = -\varepsilon_{ijk} q_k \tag{41}$$

$$\left(\delta_i{}^- q_j{}^*\right) - \left(\delta_j{}^- q_i{}^*\right) = -\varepsilon_{ijk} q_k{}^* \tag{42}$$

$$\left(\delta_i{}^+ q_j{}^*\right) - \left(\delta_j{}^- q_i\right) = 0 \tag{43}$$

These equations as well as the general commutator equations can be satisfied if q_j and q_j^* transform under a (1/2,0) and (0,1/2) representations of $SL(2,C)$. Such realizations have been discussed in Dalton (1980b). From there we have the following explicit expression for the (1/2,0) representation:

$$\left(\delta_i q_j\right) = \left[+q_4\delta_{ij} - \varepsilon_{ijk} q_k\right]/2 \tag{44}$$

$$\left(\delta_i q_j\right) = 0 \tag{45}$$

Here, $q_4 = (D^2 - q_i q_i)^{1/2}$ where D is an $SL(2,C)$ invariant. The function q_4 transforms as follows:

$$\left(\delta_i{}^+ q_4\right) = -q_i/2 \tag{46}$$

Using (44) and (46) we can easily evaluate the Casimir invariants. We have

$$-(\delta_i^+(\delta_i^+ q_u)) = \tfrac{3}{4} q_u$$
$$= \tfrac{1}{2}(\tfrac{1}{2}+1) q_u \tag{47}$$

$$-(\delta_i^-(\delta_i^- q_u)) = 0 \tag{48}$$

where q_u represents q_1 through q_4. Likewise for the (0,1/2) representation we have

$$(\delta_i^+ q_j^*) = 0 \tag{49}$$

$$(\delta_i^- q_j^*) = \left[+q_4^*\delta_{ij} - \varepsilon_{ijk}q_k^*\right]/2 \tag{50}$$

The (1/2,0) and (0,1/2) representations are the basic spinor building blocks. In the above realizations there is however an additional feature. The q_j and q_j^* have the same indices as the generators of the group.

We now give a relationship between the q_j and q_j^* and an arbitrary four component spinor. In the Pauli metric notation that we are using the Dirac matrices γ_μ satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu = 2\delta_{\mu\nu} \tag{51}$$

Consider a four-component spinor χ which transforms as follows:

$$(\delta_i\chi) = -J_i\chi = -\left(\tfrac{1}{4}\varepsilon_{ijk}\gamma_j\gamma_k\right)\chi \tag{52}$$

$$(\hat{\delta}_i\chi) = -K_i\chi = -\left(-\frac{i}{2}\gamma_4\gamma_i\right)\chi \tag{53}$$

Here J_i and K_i are matrices defined in the second equations of (28) and (29). With (51) one can show that these matrices satisfy the commutation relations for the Lie algebra in the first basis of the Appendix.

In the second basis for the Lie algebra we have

$$(\delta_i^\pm\chi) = -T_i^\pm\chi = -\tfrac{1}{2}(J_i \pm iK_i)\chi \tag{54}$$

Using (51) with J_i and K_i as defined in (52) and (53) we have

$$(\delta_i^{\pm}\chi) = iK_i\frac{\gamma_5 \pm 1}{2}\chi$$
$$= \pm iK_i P^{\pm}\chi$$
$$= J_i P^{\pm}\chi \tag{55}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, and $P^{\pm} = (1 \pm \gamma_5)/2$ are the usual projection operators with $(P^{\pm})^2 = P^{\pm}$ and $P^+P^- = 0$.

Let λ_u, $u = 1$–4 be arbitrary complex $SL(2, C)$ scalars [that is, $(\delta_i\lambda_u) = 0$, $i = 1, \ldots, 6$], and consider the following functions:

$$q_i = \lambda_u(T_i^+)^{up}\chi^p = \lambda^T T_i^+\chi \tag{56}$$

$$q_4 = \lambda^T P^+\chi/2 \tag{57}$$

where λ is a column matrix with components λ_u and $T \equiv$ transpose. We can write

$$(\delta_i^+ q_j) = \lambda^T T_j^+(\delta_i\chi) = -\lambda^T T_j^+ T_i^+\chi$$
$$= -\lambda^T J_j J_i P^+\chi \tag{58}$$

If we use the definitions of J_i in (52) with (51) we can show that the expression for $(\delta_i^+ q_j)$ in (58) is equal to the expression given in (44). Using the property $P^+P^- = 0$ we can also show that (45) holds. Likewise, we have the relations

$$(\delta_i^+ q_4) = \lambda^T P^+(\delta_i^+\chi)/2 = \lambda^T P^+(-T_i^+\chi)/2$$
$$= -\lambda^T T_i^+\chi/2 = -q_i/2 \tag{59}$$

so that (46) is satisfied. Because of the parameter relation $(\alpha_i^+)^* = \alpha_i^-$ we have in general the following relations:

$$(\delta_i^- q_u^*) = (\delta_i^+ q_u)^* \tag{60}$$

so that if q_u transforms under a $(1/2, 0)$ representation then q_u^* transforms under a $(0, 1/2)$ representation.

In the above we have show that spinor solutions of (23) can be found which satisfy the real condition (15). We have also shown that these spinor solutions can be expressed in turns of an arbitrary four-component complex spinor via (56) and (57).

With this spinor solution the transformed velocity components $\dot{x}_u'$ are now functions of a complex spinor χ, the functions ϕ_i and components $\dot{x}_u$. If either the spinor χ, or the six functions ϕ_i vanish then the transformations on the $\dot{x}_u$ reduce to the usual linear ones. The fact that the geometric quantity $\dot{x}_u\dot{x}_u$ remains invariant under transformations which mix the $\dot{x}_u$ with spinor components is of physical interest, especially in light of the fact that a variety of elementary particles are spinors.

In Dalton (1982a) it was shown that a spinor realization extended to include a nonlinear component generated the interaction of minimal coupling. The potential components A_u were related to solutions of equations identical to (41) and (43) for six functions [indicated here by f_i and $\hat{f}_i$ (or f^+ and f^-) to avoid conflict in notation]. We now consider the solution of (41) which involves these extended realizations. For this we consider the following form:

$$q_j = \lambda^T\left[f_j^{+}1 + T_j^{+}\right]\chi \tag{61}$$

where 1 is the 4×4 unit matrix, the $T_i^{\pm}$ are given in (54) and χ is a four-component column matrix. Using (61) in (41) we obtain the following relations:

$$\lambda^T\left[f_j^{+}1+T_j^{+}\right](\delta_i^{+}\chi)-\lambda^T\left[f_i^{+}1+T_i^{+}\right](\delta_j^{+}\chi)=C_{ijk}\lambda^T\left[T_k^{+}\chi\right] \tag{62}$$

To reduce to (62) we have used the above assumption that the f_j^{+} satisfy (41)–(43). The only unknown in (62) is $(\delta_i^{+}\chi)$. Equation (62) is satisfied if $(\delta_i^{+}\chi)$ is given by

$$(\delta_i^{+}\chi) = -(f_i1 + T_i^{+})\chi \tag{63}$$

Equation (63) is just the extended realization studies in Dalton (1982a). This result is interesting because it shows consistency between the nonlinear geometric transformations studied here and the nonlinear transformations of Dalton (1982a) which were related there to the minimal coupling interaction.

4. COVARIANT FORCE EQUATIONS

In this section we give a brief discussion of those acceleration equations that are covariant under these nonlinear realizations. To begin, consider the following equation:

$$\ddot{x}_\mu = \frac{d\dot{x}_\mu}{ds} = 0 \tag{64}$$

Under an infinitesimal nonlinear transformation we have

$$\ddot{x}'_\mu \equiv \frac{d}{ds}\left(\dot{x}'_\mu\right) = \frac{d}{ds}\left(\dot{x}_\mu - \alpha_i M_i^{\mu\rho}\dot{x}_\rho\right)$$

$$= \ddot{x}_\mu - \alpha_i\left(\dot{M}_i^{\mu\rho}\dot{x}_\rho + M_i^{\mu\rho}\ddot{x}_\rho\right) \tag{65}$$

Using (64) we see that

$$\ddot{x}'_\mu = \alpha_i \dot{M}_i^{\mu\rho}\dot{x}_\rho \tag{66}$$

which for arbitrary $\dot{x}_\rho$ is zero if and only if $M_i^{\mu\rho}$ is constant along the world line of the particle. For the nonlinear realizations $M_i^{\mu\rho}$ will depend on the fields that cause the nonlinearity in the transformation of the $\dot{x}_\mu$. There is no reason to expect in general that these should be constant along the world line of the particle so that $\ddot{x}_\mu = 0$ is not in general a covariant equation. If the nonlinearity in the transformation of the $\dot{x}_\mu$ is either zero, or constant along the world line of the particle then the zero acceleration condition $\ddot{x}_\mu = 0$ is covariant. This is why we have emphasized in Sections 2 and 3 those realizations which can possibly reduce to linear realizations in some limit. These solutions can represent particles which can be free ($\ddot{x}_\mu = 0$) in some limit. We point out however that there exist solutions of (12) for which the realizations cannot reduce to the linear case in any limit. Examples of this type have been published elsewhere (Dalton, 1980a). The possibility that such solutions could represent confined particles such as quarks is especially interesting, but outside the scope of this paper.

To find equations involving $\ddot{x}_\mu$ that are covariant, we construct a covariant acceleration A as follows:

$$A = \ddot{x} - F\dot{x} \tag{67}$$

In this expression $\dot{x}$, $\ddot{x}$, and A are column matrices with four components each, and F is a four-by-four matrix. We require A to transform like $\dot{x}$. If for group parameter α we have

$$\dot{x}' = N\dot{x} \tag{68}$$

then A transforms as follows:

$$A' = NA = N(\ddot{x} - F\dot{x}) \tag{69}$$

Using the form (67) for A' also, we have

$$\frac{d}{ds}(N\dot{x}) - F'N\dot{x} = N\ddot{x} - NF\dot{x} \tag{70}$$

From this equation we get the following expected relation:

$$F' = NFN^{-1} + \frac{dN}{ds} N^{-1} \tag{71}$$

Equation (71) described the transformation of the connection matrix F.

Although it is the same in form as that for guage transformations of the second kind, there is an important difference. In (71) the dependence on the world line arises through the dependence of N on the fields causing the nonlinearity. By contrast, for a gauge group, the dependence of N on the world line would arise through the group parameters.

With (71) and (69) we can see that the equation $A = 0$ implies that $A' = 0$ so that the equation

$$\ddot{x}_\mu = F^{\mu\rho} \dot{x}_\rho \tag{72}$$

is a covariant equation. This equation has the form of the traditional acceleration equation of electrodynamics. It is interesting to notice that equation (72) is invariant under both linear and nonlinear realizations of the space-type symmetry group. The equations

$$F_{\mu\rho} = 0, \qquad \ddot{x}_\mu = 0$$

are not covariant unless the nonlinearity vanishes. For this reason we see that the nonlinearity in the transformation necessitates nonzero interactions. This does not say however that forces necessitate nonlinearity because, as mentioned above, (72) is also covariant under linear transformations. We can only say that nonlinearity of the type considered here gives a reason for, and in fact necessitates forces.

The fact that the traditional force equations of electrodynamics are covariant under these nonlinear transformations would seem to suggest that the nonlinearity in these realizations is associated with electrodynamics. This may indeed be the case for some solutions but the variety of type of solutions indicated in Section 3 makes it unlikely that all of these realizations are associated with electrodynamics as traditionally recognized. For instance, as we mentioned in Section 3, one special nonlinear transforming solution has already been associated with electric charge. This leaves open the question as to the physical interpretation of the spinor, as well as other solutions.

5. COVARIANCE AND MEASUREMENT

All physical measurements, by nature, involve interactions of particles. With the idea that different types of particles are associated with different nonlinear realizations of the space-time symmetry group, we must readdress measurement and the impact of this idea on it.

Extensions to nonlinear transformations of the type considered in this paper affect measurements in two ways, and we briefly discuss them in turn. First, suppose we let $\dot{x}^T(s) = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)$, ($T$ = transpose) represent the four-velocity of a particle at point s on the world line of the particle. Under a nonlinear transformation with group parameter α we have

$$\dot{x}'(s) = N_\alpha(x(s))\dot{x}(s) \tag{73}$$

Here, the matrix $N_\alpha(x(s))$ generally depends on the position of the particle through the spinor or other fields that characterize the particle. The four-velocity $\dot{x}''(s)$ obtained from $\dot{x}(s)$ by the linear transformation

$$\dot{x}''(s) = L_\alpha \dot{x}(s) \tag{74}$$

coincides with $\dot{x}'(s)$ given in (73) only when N_α reduces to L_α. Recall from Section 4 that this could represent the free particle limit.

We have a similar situation with the space-time trajectory of the particle. Suppose we integrate (72) to obtain the space-time vector $x(s)$, where $x(s)^T = (x_1, x_2, x_3, x_4)$, as a function of s. Now (72) is covariant under both nonlinear and linear transformations. Suppose we integrate (72) after we make a *nonlinear* transformation with group parameter α. We obtain solutions $x'(s)$ along the world line. If on the other hand we first make a *linear* transformation and then integrate (72) we find that we obtain solutions $x''(s)$ which coincide with $x'(s)$ only when the nonlinearity vanishes. For instance, if the spinors, or other functions, causing the nonlinearity are oscillatory functions, the points $x'(s)$ and $x''(s)$ will coincide on a periodic basis. Now both x' and x'' are in R^4. The fact that they are not identical is simply the fact that submanifolds of R^4 (or V^4) connected by different realizations, for the same group parameters, do not coincide.

On the macroscopic level we know that the assumption of linear transformations is a good one. This does not mean however that we must have linearity on the microscopic level. If we do have nonlinearity realized on the microscopic level then it must average to zero for materials of macroscopic dimensions. One is reminded here of the domains of quantum and classical mechanics and correspondence between the two. There is much more to this similarity. One may notice that spinors solutions are obtained in these nonlinear realizations and spinors also play a key role in quantum theory of elementary particles. With macroscopic experimental apparatus, both covariant and deterministic measurements of trajectories of particles whose four-velocity transforms nonlinearly are impossible. This is because exact measurements in one frame will disagree in another since the macroscopic apparatus transforms linearly.

In addition to the above situation, there is a second measurement feature of equal interest. If we have two solutions of (23) then their superposition is also a solution. With this, the nonlinearity in the realizations for a given particle could depend upon all of the fields in the neighborhood. If this is the case, one would expect that the nonlinearity to have a larger effect for interacting particles. However, if superposition does hold, the field of a probe particle will affect the nonlinearity of the particle one wishes to detect. This would make it impossible to exactly measure the velocities and positions of a particle since the very act of measurement changes the way they transform. Again we are reminded of a well-known measurement problem of quantum mechanics.

In conclusion we notice that both Lorentz forces and spinors play a fundamental role in physics. If we limit our theories to include only linear transformations we must include these as additional input. However, if we include those nonlinear realizations that leave $\dot{x}_\mu \dot{x}_\mu$ invariant we involve the Lorentz forces as well as spinors in a natural way. In addition, the interesting measurement aspects that arise when one includes the nonlinear transformations have an uncanny similarity to measurement aspects of quantum theory. If these nonlinear realizations are indeed "realized" in nature, the idea that the nature of matter is essentially geometric will have much support.

APPENDIX: LIE ALGEBRA BASIS

For use in the present and future papers we briefly review here in our notation the commutation relations of the Lie algebra for $SL(2,C)$ in two convenient bases.

In the first basis we use three real parameters ω_i ($i=1$–3) for the $SU(2)$ subgroup and the three real parameters ν_i ($i=1,2,3$) for the pure Lorentz boost. The infinitesimal transformation on an arbitrary variable ε is given as follows:

$$\varepsilon' = \varepsilon + \omega_i(\delta_i\varepsilon) + \nu_i(\delta_i\varepsilon) \tag{A1}$$

where the sum on i is from 1 to 3.

As in Dalton (1980b) we have chosen to absorb the usual factor of $(-1)^{1/2}$ into the $(\delta_i\varepsilon)$ and $(\hat{\delta}_i\varepsilon)$. The expressions $(\delta_i\varepsilon)$ and $(\hat{\delta}_i\varepsilon)$ are defined by (2) where the derivatives are with respect to the real parameters ω_i and

ν_i, respectively. With this basis the commutator relations are expressed as follows:

$$(\delta_i(\delta_j\varepsilon)) - (\delta_j(\delta_i\varepsilon)) = -\varepsilon_{ijk}(\delta_k\varepsilon) \tag{A2}$$

$$(\delta_i(\hat{\delta}_j\varepsilon)) - (\hat{\delta}_j(\delta_i\varepsilon)) = -\varepsilon_{ijk}(\hat{\delta}_k\varepsilon) \tag{A3}$$

$$(\hat{\delta}_i(\hat{\delta}_j\varepsilon)) - (\hat{\delta}_j(\hat{\delta}_i\varepsilon)) = -\varepsilon_{ijk}(\delta_k\varepsilon) \tag{A4}$$

where ε_{ijk} is the total antisymmetric tensor in three indices with $\varepsilon_{123} = +1$.

To compare with and take advantage of previous classifications of realizations of $SL(2,C)$ we consider the following basis:

$$(\delta_i^{+}\varepsilon) \equiv [(\delta_i\varepsilon) + i(\hat{\delta}_i\varepsilon)]/2 \tag{A5}$$

$$(\delta_i^{-}\varepsilon) \equiv [(\delta_i\varepsilon) - i(\hat{\delta}_i\varepsilon)]/2 \tag{A6}$$

We have the inverse relations

$$(\delta_i\varepsilon) = (\delta_i^{+}\varepsilon) + (\delta_i^{-}\varepsilon) \tag{A7}$$

$$(\hat{\delta}_i\varepsilon) = -i[(\delta_i^{+}\varepsilon) - (\delta_i^{-}\varepsilon)] \tag{A8}$$

In this basis we have

$$\varepsilon' = \varepsilon + \alpha_i^{+}(\delta_i^{+}\varepsilon) + \alpha_i^{-}(\delta_i^{-}\varepsilon) \tag{A9}$$

where $\alpha_i^{+} = \omega_i + i\nu_i$ and $\alpha_i = \omega_i - i\nu_i$. Using (A5) and (A6) in (A2)–(A4) we arrive at the following set of commutation relations:

$$(\delta_i^{+}(\delta_j^{+}\varepsilon)) - (\delta_j^{+}(\delta_i^{+}\varepsilon)) = -\varepsilon_{ijk}^{+}(\delta_k\varepsilon) \tag{A10}$$

$$(\delta_i^{-}(\delta_j^{-}\varepsilon)) - (\delta_j^{-}(\delta_i^{-}\varepsilon)) = -\varepsilon_{ijk}(\delta_k^{-}\varepsilon) \tag{A11}$$

$$(\delta_i^{-}(\delta_j^{+}\varepsilon)) - (\delta_j^{+}(\delta_i^{-}\varepsilon)) = 0 \tag{A12}$$

This basis corresponds to a direct product decomposition of $SL(2,C)$ into two $SU(2)$-type subgroups but each with complex rather than real parameters. From Dalton (1980b) the two Casimir invariants are given by

$$-(\delta_i^{+}(\delta_i^{+}\varepsilon)) = +j_1(j_1+1)\varepsilon \tag{A13}$$

$$-(\delta_i^{-}(\delta_i^{-}\varepsilon)) = j_2(j_2+1)\varepsilon \tag{A14}$$

where j_1 and j_2 are numbers. The negative sign in the above two equations is included because as mentioned before we absorbed the usual factor of $(-1)^{1/2}$ into the expressions $(\delta_i \varepsilon)$. Different realizations of $SL(2, C)$ may be partially classified by giving the set of numbers (j_1, j_2). We emphasize however that knowing (j_1, j_2) does *not* tell us whether or not the realization is linear or nonlinear.

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